

# RELATIVE SYMMETRIC POLYNOMIALS AND MONEY CHANGE PROBLEM

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**ABSTRACT.** This article is devoted to the number of non-negative solutions of the linear Diophantine equation

$$a_1t_1 + a_2t_2 + \cdots + a_nt_n = d,$$

where  $a_1, \dots, a_n$ , and  $d$  are positive integers. We obtain a relation between the number of solutions of this equation and characters of the symmetric group, using *relative symmetric polynomials*. As an application, we give a necessary and sufficient condition for the space of the relative symmetric polynomials to be non-zero.

**AMS Subject Classification** Primary 05A17, Secondary 05E05 and 15A69.

**Key Words** Money change problem; Partitions of integers; Relative symmetric polynomials; Symmetric groups; Complex characters.

Suppose  $a_1, \dots, a_n$ , and  $d$  are positive integers, and consider the following linear Diophantine equation:

$$a_1t_1 + a_2t_2 + \cdots + a_nt_n = d.$$

Let  $Q_d(a_1, \dots, a_n)$  be the number of non-negative integer solutions of this equation. Computing the exact values of the function  $Q_d$  is the well-known *money change problem*. It is easy to see that a generating function for  $Q_d(a_1, \dots, a_n)$  is

$$\prod_{i=1}^n \frac{1}{1 - t^{a_i}}.$$

This article is devoted for an interesting relation between  $Q_d$  and irreducible complex characters of the symmetric group  $S_m$ , where  $m = a_1 + \cdots + a_n$ . In fact, we will show that  $Q_d$  is a permutation character of  $S_m$ , and then we will find its irreducible constituents. Our main tool, in the investigation of  $Q_d$ , is the notion of *relative symmetric polynomials*, which is introduced by the author in [3]. Once, we find the irreducible constituents of  $Q_d$ , we can also obtain a necessary and sufficient condition for vanishing of the space

of relative symmetric polynomials. A similar result was obtained in [2], for vanishing of *symmetry classes of tensors*, using the same method.

We need a survey of results about relative symmetric polynomials in this article. For a detailed exposition, one can see [3].

Let  $G$  be a subgroup of the full symmetric group  $S_m$  of degree  $m$  and suppose  $\chi$  is an irreducible complex character of  $G$ . Let  $H_d[x_1, \dots, x_m]$  be the complex space of homogenous polynomials of degree  $d$  with the independent commuting variables  $x_1, \dots, x_m$ . Suppose  $\Gamma_{m,d}^+$  is the set of all  $m$ -tuples of non-negative integers,  $\alpha = (\alpha_1, \dots, \alpha_m)$ , such that  $\sum_i \alpha_i = d$ . For any  $\alpha \in \Gamma_{m,d}^+$ , define  $X^\alpha$  to be the monomial  $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ . So the set  $\{X^\alpha : \alpha \in \Gamma_{m,d}^+\}$  is a basis of  $H_d[x_1, \dots, x_m]$ . We define also an inner product on  $H_d[x_1, \dots, x_m]$  by

$$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta}.$$

The group  $G$  acts on  $H_d[x_1, \dots, x_m]$  via

$$q^\sigma(x_1, \dots, x_m) = q(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(m)}).$$

It also acts on  $\Gamma_{m,d}^+$  by

$$\sigma\alpha = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}).$$

Let  $\Delta$  be a set of representatives of orbits of  $\Gamma_{m,d}^+$  under the action of  $G$ .

Now consider the idempotent

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \sigma$$

in the group algebra  $\mathbb{C}G$ . Define the space of *relative symmetric polynomials of degree  $d$*  with respect to  $G$  and  $\chi$  to be

$$H_d(G, \chi) = T(G, \chi)(H_d[x_1, \dots, x_m]).$$

Let  $q \in H_d[x_1, \dots, x_m]$ . Then we set

$$q^* = T(G, \chi)(q)$$

and we call it a *symmetrized polynomial* with respect to  $G$  and  $\chi$ . Clearly

$$H_d(G, \chi) = \langle X^{\alpha,*} : \alpha \in \Gamma_{m,d}^+ \rangle,$$

where  $\langle \text{set of vectors} \rangle$  denotes the subspace generated by a given set of vectors.

Recall that the inner product of two characters of an arbitrary group  $K$  is defined as follows,

$$[\phi, \psi]_K = \frac{1}{|K|} \sum_{\sigma \in K} \phi(\sigma) \psi(\sigma^{-1}).$$

In the special case where  $K$  is a subgroup of  $G$  and  $\phi$  and  $\psi$  are characters of  $G$ , the notation  $[\phi, \psi]_K$  will denote the inner product of the restrictions of  $\phi$  and  $\psi$  to  $K$ .

It is proved in [3] that for any  $\alpha$ , we have

$$\|X^{\alpha,*}\|^2 = \chi(1) \frac{[\chi, 1]_{G_\alpha}}{[G : G_\alpha]},$$

where  $G_\alpha$  is the stabilizer subgroup of  $\alpha$  under the action of  $G$ . Hence,  $X^{\alpha,*} \neq 0$ , if and only if  $[\chi, 1]_{G_\alpha} \neq 0$ . According to this result, let  $\Omega$  be the set of all  $\alpha \in \Gamma_{m,d}^+$ , with  $[\chi, 1]_{G_\alpha} \neq 0$  and suppose  $\bar{\Delta} = \Delta \cap \Omega$ .

We proved in [3], the following formula for the dimension of  $H_d(G, \chi)$

$$\dim H_d(G, \chi) = \chi(1) \sum_{\alpha \in \bar{\Delta}} [\chi, 1]_{G_\alpha}.$$

Note that,  $\bar{\Delta}$  depends on  $\chi$ , but  $\Delta$  depends only on  $G$ . Since,  $[\chi, 1]_{G_\alpha} = 0$ , for all  $\alpha \in \Delta - \bar{\Delta}$ , we can re-write the above formula, as

$$\dim H_d(G, \chi) = \chi(1) \sum_{\alpha \in \Delta} [\chi, 1]_{G_\alpha}.$$

There is also another interesting formula for the dimension of  $H_d(G, \chi)$ . This is the formula which employs the function  $Q_d$  and so it connects the money change problem to relative symmetric polynomials. Let  $\sigma \in G$  be any element with the cycle structure  $[a_1, \dots, a_n]$ , (i.e.  $\sigma$  is equal to a product of  $n$  disjoint cycles of length  $a_1, \dots, a_n$ , respectively). Define  $Q_d(\sigma)$  to be the number of non-negative integer solutions of the equation

$$a_1 t_1 + a_2 t_2 + \dots + a_n t_n = d,$$

so, we have  $Q_d(\sigma) = Q_d(a_1, \dots, a_n)$ . If we consider the free vector space  $\mathbb{C}[\Gamma_{m,d}^+]$  as a  $\mathbb{C}G$ -module, then for all  $\sigma \in G$ , we have

$$Tr \sigma = Q_d(\sigma),$$

and hence,  $Q_d$  is a permutation character of  $G$ . It is proved in [3], that we have also

$$\dim H_d(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q_d(\sigma).$$

Note that, we can write this result as  $\dim H_d(G, \chi) = \chi(1) [\chi, Q_d]_G$ . Now, comparing two formulae for the dimension of  $H_d(G, \chi)$  and using the *reciprocity relation* for induced characters, we obtain

$$Q_d(a_1, \dots, a_n) = \sum_{\alpha \in \Delta} (1_{G_\alpha})^G(\sigma),$$

where  $\sigma \in S_m$  is any permutation of the cycle structure  $[a_1, \dots, a_n]$ ,  $G$  is any subgroup of  $S_m$  containing  $\sigma$  and  $m = a_1 + \dots + a_n$ . It is clear that, if  $\alpha$  and  $\beta$  are in the same orbit of  $\Gamma_{m,d}^+$ , then  $(1_{G_\alpha})^G = (1_{G_\beta})^G$ , so we have also

$$Q_d(a_1, \dots, a_n) = \frac{1}{|G|} \sum_{\alpha \in \Gamma_{m,d}^+} |G_\alpha| (1_{G_\alpha})^G(\sigma).$$

As our main result in this section, we have,

**Theorem A.**

$$Q_d = \frac{1}{|G|} \sum_{\alpha \in \Gamma_{m,d}^+} |G_\alpha| (1_{G_\alpha})^G.$$

In the remaining part of this article, we will assume that,  $G = S_m$ , then using representation theory of symmetric groups, we will find, irreducible constituents of  $Q_d$ .

We need some standard notions from representation theory of symmetric groups. Ordinary representations of  $S_m$  are in one to one correspondence with *partitions* of  $m$ . Let

$$\pi = (\pi_1, \dots, \pi_s)$$

be any partition of  $m$ . The irreducible character of  $S_m$ , corresponding to a partition  $\pi$  is denoted by  $\chi^\pi$ . There is also a subgroup of  $S_m$ , associated to  $\pi$ , which is called the *Young subgroup* and it is defined as,

$$S_\pi = S_{\{1, \dots, \pi_1\}} \times S_{\{\pi_1+1, \dots, \pi_1+\pi_2\}} \times \dots$$

Therefore, we have  $S_\pi \cong S_{\pi_1} \times \dots \times S_{\pi_s}$ .

Let  $\pi = (\pi_1, \dots, \pi_s)$  and  $\mu = (\mu_1, \dots, \mu_l)$  be two partitions of  $m$ . We say that  $\mu$  majorizes  $\pi$ , iff for any  $1 \leq i \leq \min\{s, l\}$ , the inequality

$$\pi_1 + \dots + \pi_i \leq \mu_1 + \dots + \mu_i$$

holds. In this case we write  $\lambda \leq \mu$ . This is clearly a partial ordering on the set of all partitions of  $m$ . A generalized  $\mu$ -tableau of type  $\pi$  is a function

$$T : \{(i, j) : 1 \leq i \leq h(\mu), 1 \leq j \leq \mu_i\} \rightarrow \{1, 2, \dots, m\}$$

such that for any  $1 \leq i \leq m$ , we have  $|t^{-1}(i)| = \pi_i$ . This generalized tableau is called semi-standard if for each  $i$ ,  $j_1 < j_2$  implies  $T(i, j_1) \leq T(i, j_2)$  and for any  $j$ ,  $i_1 < i_2$  implies  $T(i_1, j) < T(i_2, j)$ . In other words,  $T$  is semi-standard, iff every row of  $T$  is non-descending and every column of  $T$  is ascending. The number of all such semi-standard tableaux is denoted by  $K_{\mu\pi}$  and it is called the *Kostka number*. It is well known that  $K_{\mu\pi} \neq 0$  iff  $\mu$  majorizes  $\pi$ , see [1], for example. We have also,

$$\begin{aligned} K_{\mu\pi} &= [(1_{S_\pi})^{S_m}, \chi^\mu]_{S_m} \\ &= [1, \chi^\mu]_{S_\pi}. \end{aligned}$$

For any  $\alpha \in \Gamma_{m,d}^+$ , the multiplicity partition is denoted by  $M(\alpha)$ , so, to obtain  $M(\alpha)$ , we must arrange the multiplicities of numbers  $0, 1, \dots, d$  in  $\alpha$  in the descending order. It is clear that  $(S_m)_\alpha \cong S_{M(\alpha)}$ , the Young subgroup. If  $M(\alpha) = (k_1, \dots, k_s)$ , then we have

$$|(S_m)_\alpha| = k_1! k_2! \dots k_s!.$$

In what follows, we use the notation  $M(\alpha)!$  for the number  $k_1!k_2!\dots k_s!$ . On the other hand, we have

$$\begin{aligned} (1_{G_\alpha})^G &= (1_{S_{M(\alpha)}})^{S_m} \\ &= \sum_{M(\alpha) \leq \pi} K_{\pi, M(\alpha)} \chi^\pi. \end{aligned}$$

Now, using Theorem A, we obtain,

**Theorem B.**

$$Q_d = \frac{1}{m!} \sum_{\alpha \in \Gamma_{m,d}^+} \sum_{M(\alpha) \leq \pi} M(\alpha)! K_{\pi, M(\alpha)} \chi^\pi.$$

As a result, we can compute the dimension of  $H_d(S_m, \chi^\pi)$ , in a new fashion. We have,

$$\begin{aligned} \dim H_d(S_m, \chi^\pi) &= \chi^\pi(1) [\chi^\pi, Q_d]_{S_m} \\ &= \frac{\chi^\pi(1)}{m!} \sum_{\alpha \in \Gamma_{m,d}^+, M(\alpha) \leq \pi} M(\alpha)! K_{\pi, M(\alpha)}. \end{aligned}$$

Note that, this generalizes the similar formulae in the final part of the second section of [3]. Now, as a final result, we have also, a necessary and sufficient condition for  $H_d(S_m, \chi^\pi)$  to be non-zero.

**Theorem C.**

We have  $H_d(S_m, \chi^\pi) \neq 0$ , if and only if there exists  $\alpha \in \Gamma_{m,d}^+$ , such that  $M(\alpha) \leq \pi$ .

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